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TEMPERATURE DISTRIBUTION IN CHANNEL FLOW WITH FRICTION

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Abstract—The usually neglected pressure drop in the energy equation must be taken into account if the heat of friction is not negligible. The thus obtained temperature distribution differs from that which is usually found in literature. For laminar flow an effect of temperature drop, similar to the Ranque effect, is to be observed.

NOMENCLATURE

a, diffusivity of heat;
c_p, specific heat at constant pressure;
g', gravity acceleration in the technical system of units and $g'=1$ in physical system of units;
p, pressure;
(Pr), Prandtl number;
T, temperature of the fluid;
T_w, wall temperature;
T_0, initial temperature of the fluid;
T_{min}, temperature in tube centre;
w_z, velocity;
w, maximum velocity;
w_m, mean velocity;
x, y, z, Cartesian co-ordinates;
$y^+ = y/R$, dimensionless co-ordinate; R—half a distance between parallel walls of the flat conduit, or tube radius.

Greek symbols

γ, specific gravity (or specific density in physical system of units);
θ, dimensionless variable, equation (11);
λ, heat conductivity;
μ, ν, dynamic and kinematic viscosities;
ϕ, dissipation function;
τ, time.

1. INTRODUCTION

THE energy equation

$$c_p\gamma \frac{\mathrm{d}T}{\mathrm{d}\tau} = \frac{\mathrm{d}p}{\mathrm{d}\tau} + \mu\phi + \nabla(\lambda\nabla T) \tag{1}$$

is usually solved for cases, where the term

$$f = \frac{\mathrm{d}p}{\mathrm{d}\tau} + \mu\phi \tag{2}$$

may be neglected because of small pressure drop and small values of frictional heat. If, however, the heat of friction is taken into account, one cannot simultaneously neglect the term $\mathrm{d}p/\mathrm{d}\tau$, as it is to be found in the literature (cf. [1], [2], [3] etc.) since, at least for laminar channel flow, both the terms in (2) are of the same order. Let z be the axis of a straight channel of constant cross section, and $w_z(x, y)$ the velocity. If the flow is laminar and stationary, from equation of motion we get

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \frac{\partial}{\partial x}\left(\mu\frac{\partial w_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w_z}{\partial y}\right). \tag{3}$$

Since

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = w_z\frac{\mathrm{d}p}{\mathrm{d}z} = w_z\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial w_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w_z}{\partial y}\right)\right]$$

and

$$\phi = \mu\left[\left(\frac{\partial w_z}{\partial x}\right)^2 + \left(\frac{\partial w_z}{\partial y}\right)^2\right]$$

we obtain by substitution into (2)

$$f = w_z\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial w_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial w_z}{\partial y}\right)\right] + \mu\left[\left(\frac{\partial w_z}{\partial x}\right)^2 + \left(\frac{\partial w_z}{\partial y}\right)^2\right]$$

$$= \frac{\partial}{\partial x}\left(\mu w_z\frac{\partial w_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu w_z\frac{\partial w_z}{\partial y}\right)$$

$$= \nabla\left(\mu\nabla\frac{w_z^2}{2}\right). \qquad (4)$$

Introduction of (4) into (1) yields

$$c_p\gamma\frac{\mathrm{d}T}{\mathrm{d}\tau} = c_p\gamma w_z\frac{\partial T}{\partial z} = \nabla\left(\mu\nabla\frac{w_z^2}{2} + \lambda\nabla T\right) \qquad (5)$$

or, for fluid with constant properties

$$w_z\frac{\partial T}{\partial z} = a\nabla^2\left[(Pr)\frac{w_z^2}{2g'c_p} + T\right]. \qquad (6)$$

This equation has a particular solution

$$\nabla\left[(Pr)\frac{w_z^2}{2g'c_p} + T\right] = \text{constant}, \qquad (7)$$

if the temperature profile is accomplished (i.e. it does not depend upon z as well as the velocity w_z). For channels that solution is

$$(Pr)\frac{w_z^2}{2g'c_p} + T = T_w, \qquad (8)$$

and is equal to the temperature distribution in the vortex tube. A similar effect, as in the Ranque–Hilsch tube, may be thus expected in laminar channel flow.

2. TEMPERATURE DISTRIBUTION IN POISEUILLE FLOW

For round tubes or flat conduits it is

$$w_z = w(1 - y^{+2}) \qquad (9)$$

where w is the maximum velocity (velocity in the axis). Equation (7) yields

$$\frac{\mathrm{d}}{\mathrm{d}y^+}[(1 - y^{+2})^2 + \theta] = \text{constant} \qquad (10)$$

with

$$\theta = \frac{2g'c_p\,T}{(Pr)\,.\,w^2}. \qquad (11)$$

The constant (10) vanishes because of symmetry, wherefore

$$(1 - y^{+2})^2 + \theta = \text{constant} = \theta_w \qquad (12)$$

where $\theta_w = \theta(1)$. This relation is shown in Fig. 1.

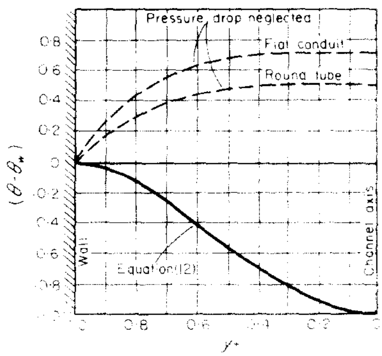

FIG. 1.

Differentiating (12) we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}y^+} = 2y^+(1 - y^{+2})$$

and

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}y^+}\right)_{y^+=1} = 0$$

so that the wall is adiabatic.

(Fig. 1 contains also temperature profiles obtained with neglection of pressure drop after [1]).

The character of these profiles is quite different, as the maximum temperature is not at the walls, but in the centre.

It follows thus that the temperature in the channel centre is lower than that at the wall in an adiabatic laminar flow. But as for the latter the rate of flow of the sum of total enthalpy and kinetic energy must be constant, it follows that

$$Fw_m\left(c_pT_0 + \frac{w_m^2}{2g'}\right) = \int_F \mathrm{d}F\,.\,w_z\left(c_pT + \frac{w_z^2}{2g'}\right) \qquad (13)$$

where T_0 is the initial (entrance) temperature, and w_m—the mean velocity.

With (11) variable and

$$\theta_0 = \frac{2g'c_pT_0}{(Pr)w^2} \qquad (14)$$

(13) yields for round tubes

$$\theta_w = \theta_0 + \frac{(Pr) - \frac{1}{2}}{2(Pr)} \tag{15}$$

and

$$\theta_{\min} = \theta_w - 1 = \theta_0 - \frac{(Pr) + \frac{1}{2}}{2(Pr)}. \tag{16}$$

The temperature increase at the wall $\Delta\theta_w = \theta_w - \theta_0$ and the temperature drop $\Delta\theta = \theta_0 - \theta_{\min}$ in the tube centre are thus functions of the Prandtl number, shown in Fig. 2.

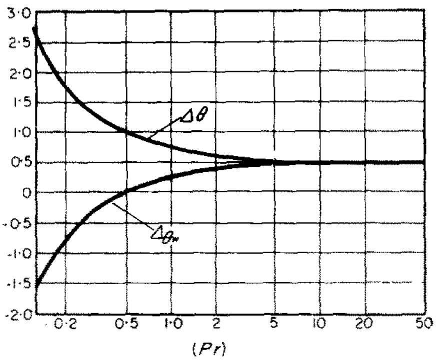

FIG. 2.

For $(Pr) > 10$ the cooling and heating effects stabilize; for $(Pr) < 0{\cdot}5$ there is no heating effect. Calculations show that for water at 20°C [$(Pr) \approx 7$] the temperature drop between the initial temperature T_0 and the minimum temperature $T_{\min}$ is equal to

$$T_0 - T_{\min} = \Delta\theta \frac{(Pr)w^2}{2g'c_p} = \frac{7{\cdot}2 \times 10^{-15}}{d^2}\ ^\circ\mathrm{C}$$

if we assume $(Re) = w_m d/\nu = 2000$. Assuming even $d = 10^{-3}$ m we obtain $T_0 - T_{\min} = 7{\cdot}2 \times 10^{-9}$ °C, so that the cooling effect is negligible. It is expected only in highly viscous fluids. For example, for pure glycerine at 20°C with $(Pr) = 8774$ the temperature drop is 6°C even for $d = 0{\cdot}1$ m and $(Re) = 200$. For such cases a problem of tube length needed for the stabilization of temperature profile may arise. This is connected with the solution $T(x, y, z)$ of (6), satisfying the conditions $\nabla T = 0$ at the adiabatic wall, and $T = T_0$ at $z = 0$. That solution may be obtained by similar methods as are used in the Graetz problem.

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Résumé—La perte de charge que l'on néglige habituellement dans l'equation de l'énergie doit être prise en considération quand la chaleur due au frottement n'est pas négligeable. La distribution de température ainsi obtenue diffère de celle que l'on trouve habituellement dans la littérature. On observe en écoulement laminaire une chute de température semblable à celle de l'effet Ranque.

Zusammenfassung—Der gewöhnlich in der Energiegleichung vernachlässigte Druckabfall ist bei nicht vernachlässigbarer Reibungswärme zu berücksichtigen. Die dabei erhaltene Temperaturverteilung unterscheidet sich von der in der Literatur gewöhnlich angegebenen. Bei laminarer Strömung ist ein Temperaturabfall ähnlich dem des Ranque-Effekts zu beobachten.

Аннотация—Перепад давления, которым обычно пренебрегают, необходимо учитывать в уравнении энергии для потока жидкости, если теплом трения нельзя пренебречь. Распределение температуры, полученное таким образом, отличается от распределения температуры, которое обычно указавают в литературе. Для ламинарного потока влияние перепада температуры должно быть аналогично эффекту Ранка.